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Duality for a Class of Multiobjective Control Problems

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In this paper, a class of multiobjective control problems is considered, where the objective and constraint functions involved are $f(t, x(t), \dot{x}(t), y(t), z(t))$ with $x(t) \in R^n$, $y(t) \in R^n$, and $z(t) \in R^m$, where $x(t)$ and $z(t)$ are the control variables and $y(t)$ is the state variable. Under the assumption of invexity and its generalization, duality theorems are proved through a parametric approach to related properly efficient solutions of the primal and dual problems.

1. INTRODUCTION

The problem of optimal control with mixed equality and inequality constrains is considered in [6, 8]. B. Mond and I. Smart [8] established duality results for a control problem under invexity and showed that for invex functions, the necessary conditions for optimality in the control problem are also sufficient. Bhatia and Kumar [1] discussed the multiobjective control problems with ρ–pseudoinvexity, ρ–strictly pseudoinvexity, ρ–quasiinvexity, or ρ–strictly quasiinvexity. Nahak and Nanda [9] discussed the efficiency and duality for multiobjective variational control problems with (F, ρ)–convexity. The objective and constraint functions in both papers were different. Xiuhong [4, 5] also considered duality and symmetric duality for two types of multiobjective variational problems, respectively. In this paper, the objective and constraint functions considered are $f(t, x(t), \dot{x}(t), y(t), z(t))$ with $x(t) \in R^n$, $y(t) \in R^n$, and $z(t) \in R^m$, where $x(t)$ and $z(t)$ are the control variables and $y(t)$ is the state variable. Under the invexity assumption on the functions involved, duality theorems are proved through a parmetric approach to related properly efficient solutions. These results generalize the results of [2, 3, 7, 8].

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2. PRELIMINARIES AND NOTATIONS

Let $I = [a, b]$ be a real interval, $f_i(t, x(t), \dot{x}(t), y(t), z(t))$ $(i = 1, 2, \dots, p)$, $g_j(t, x(t), \dot{x}(t), y(t), z(t))$ $(j = 1, 2, \dots, q)$, and let $h_k(t, x(t), \dot{x}(t), y(t), z(t))$ $(k = 1, 2, \dots, n)$ be continuously differentiable functions, where t is the independent variable, $x : I \to R^n$ and $z : I \to R^m$ are the control variables, and $y : I \to R^n$ is the state variable. $x(t)$ and $z(t)$ are related to $y(t)$ via the state equations $h(t, x(t), \dot{x}(t), y(t), z(t)) = \dot{y}(t)$, where the dot denotes a derivative with respect to t. Denote the first partial derivatives of f_i with respect to t, x, $\dot{x}$, y, and z, respectively, by f_{it}, f_{ix}, $f_{i\dot{x}}$, f_{iy}, and f_{iz}; that is,

$$f_{it} = \frac{\partial f_i}{\partial t},$$

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \dots, \frac{\partial f_i}{\partial x_n}\right)^T, \qquad f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \frac{\partial f_i}{\partial \dot{x}_2}, \dots, \frac{\partial f_i}{\partial \dot{x}_n}\right)^T,$$

$$f_{iy} = \left(\frac{\partial f_i}{\partial y_1}, \frac{\partial f_i}{\partial y_2}, \dots, \frac{\partial f_i}{\partial y_n}\right)^T, \qquad f_{iz} = \left(\frac{\partial f_i}{\partial z_1}, \frac{\partial f_i}{\partial z_2}, \dots, \frac{\partial f_i}{\partial z_m}\right)^T,$$

$i = 1, 2, \dots, p$, where T denotes the transpose operator. Similarly, $g_{jt}, g_{jx}, g_{j\dot{x}}, g_{jy}$ and g_{jz} denote the first partial derivatives of g_j with respect to t, $x(t)$, $\dot{x}(t)$, $y(t)$, and $z(t)$ for $j = 1, 2, \dots, q$; $h_{kt}, h_{kx}, h_{k\dot{x}}, h_{ky}$, and h_{kz} denote the first partial derivatives of h_k with respect to t, $x(t)$, $\dot{x}(t)$, $y(t)$, and $z(t)$ for $k = 1, 2, \dots, n$.

For an r–dimensional vector function $R(t, x(t), \dot{x}(t), y(t), z(t))$, denote the first partial derivative with respect to t, $x(t)$, $\dot{x}(t)$, $y(t)$, and $z(t)$ by $R_t, R_x, R_{\dot{x}}, R_y$, and R_z, respectively; that is,

$$R_t = \left(\frac{\partial R^1}{\partial t}, \frac{\partial R^2}{\partial t}, \dots, \frac{\partial R^r}{\partial t}\right)^T,$$

$$R_x = \begin{pmatrix} \frac{\partial R^1}{\partial x_1} & \frac{\partial R^1}{\partial x_2} & \cdots & \frac{\partial R^1}{\partial x_n} \\ \frac{\partial R^2}{\partial x_1} & \frac{\partial R^2}{\partial x_2} & \cdots & \frac{\partial R^2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial R^r}{\partial x_1} & \frac{\partial R^r}{\partial x_2} & \cdots & \frac{\partial R^r}{\partial x_n} \end{pmatrix}_{r \times n}, \quad R_{\dot{x}} = \begin{pmatrix} \frac{\partial R^1}{\partial \dot{x}_1} & \frac{\partial R^1}{\partial \dot{x}_2} & \cdots & \frac{\partial R^1}{\partial \dot{x}_n} \\ \frac{\partial R^2}{\partial \dot{x}_1} & \frac{\partial R^2}{\partial \dot{x}_2} & \cdots & \frac{\partial R^2}{\partial \dot{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial R^r}{\partial \dot{x}_1} & \frac{\partial R^r}{\partial \dot{x}_2} & \cdots & \frac{\partial R^r}{\partial \dot{x}_n} \end{pmatrix}_{r \times n},$$

$$R_y = \begin{pmatrix} \frac{\partial R^1}{\partial y_1} & \frac{\partial R^1}{\partial y_2} & \cdots & \frac{\partial R^1}{\partial y_n} \\ \frac{\partial R^2}{\partial y_1} & \frac{\partial R^2}{\partial y_2} & \cdots & \frac{\partial R^2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial R^r}{\partial y_1} & \frac{\partial R^r}{\partial y_2} & \cdots & \frac{\partial R^r}{\partial y_n} \end{pmatrix}_{r\times n}, \quad R_z = \begin{pmatrix} \frac{\partial R^1}{\partial z_1} & \frac{\partial R^1}{\partial z_2} & \cdots & \frac{\partial R^1}{\partial z_m} \\ \frac{\partial R^2}{\partial z_1} & \frac{\partial R^2}{\partial z_2} & \cdots & \frac{\partial R^2}{\partial z_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial R^r}{\partial z_1} & \frac{\partial R^r}{\partial z_2} & \cdots & \frac{\partial R^r}{\partial z_m} \end{pmatrix}_{r\times m}.$$

Denote by X the space of piecewise smooth control functions $x : I \to R^n$ with norm $\|x\|_\infty$; Z the space of piecewise continuous control functions $z : I \to R^m$ with norm $\|z\|_\infty$; Y the space of piecewise continuous differentiable state functions $y : I \to R^n$ with norm $\|y\| = \|y\|_\infty + \|Dy\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Longleftrightarrow x(t) = u(a) + \int_a^b u(s)\,ds,$$

where $u(a)$ is a given boundary value. Therefore $\frac{d}{dt} = D$ except at discontinuities.

Define

$$\Lambda^+ = \{\tau \in \mathbf{R}^p | \tau > 0, \tau^T e = 1, e = (1, 1, \ldots, 1)^T \in \mathbf{R}^p\}.$$

Denote $\mathbf{R}^p_+$ be the non-negative orthant of $\mathbf{R}^p$.

Consider the following multiobjective control problem

$$\text{(MCP)} \quad \text{minimize} \left(\int_a^b f_1(t, x(t), \dot{x}(t), y(t), z(t))dt \ldots, \int_a^b f_p(t, x(t), \dot{x}(t), y(t), z(t))dt \right)$$

$$\text{subject to } x(a) = y(a) = z(a) = 0, \; x(b) = y(b) = z(b) = 0,$$

$$\dot{x}(a) = 0 = \dot{x}(b), \tag{1}$$

$$\dot{y}(t) = h(t, x(t), \dot{x}(t), y(t), z(t)), \forall t \in I \tag{2}$$

$$g(t, x(t), \dot{x}(t), y(t), z(t)) = (g_1(t, x(t), \dot{x}(t), y(t), z(t)), \ldots, g_q(t, x(t), \dot{x}(t), y(t), z(t)))^T \leq 0, \; \forall\, t \in I, \tag{3}$$

where the functions involved are different from those of [1, 3, 6], but are similar to the form of [5, 8].

DEFINITION 1. A point $(x^*(t), y^*(t), z^*(t))$ is said to be an efficient solution of (MCP), if for all feasible solution $(x(t), y(t), z(t))$ of (MCP)

$$\int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\,dt \geq \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt \quad i = 1, 2, \ldots, p,$$

then

$$\int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\,dt$$
$$= \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt \qquad i = 1, 2, \ldots, p.$$

The point $(x^*(t), y^*(t), z^*(t))$ is said to be a properly efficient solution of (MCP), if there exists a scalar real $M > 0$ such that for all $i \in \{1, 2, \ldots, p\}$

$$\int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))dt - \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\ dt$$
$$\leq M\left(\int_a^b f_j(t, x(t), \dot{x}(t), y(t), z(t))dt - \int_a^b f_j(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\ dt\right)$$

for some j, such that

$$\int_a^b f_j(t, x(t), \dot{x}(t), y(t), z(t))\,dt > \int_a^b f_j(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\,dt$$

whenever the feasible solution $(x(t), y(t), z(t))$ of (MCP) and

$$\int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\,dt > \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt.$$

LEMMA 1. *If $(x^*(t), y^*(t), z^*(t))$ is a properly efficient solution for (MCP), then $(x^*(t), y^*(t), z^*(t))$ solves the following problem for some $\tau \in \Lambda^+$*

$$\textit{minimize} \sum_{i=1}^{p} \tau_i \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$$
$$\textit{subject to } (1), (2), \textit{ and } (3).$$

Proof. The process is very similar to that for Lemma 2 of [4]. ∎

LEMMA 2. *If $(x^*(t), y^*(t), z^*(t))$ solves the following problem*

$$\textit{minimize} \int_a^b f(t, x(t), \dot{x}(t), y(t), z(t))\,dt$$
$$\textit{subject to } (1), (2), \textit{ and } (3).$$

If the Fréchet derivative $[D - H_y(x^*, y^*, z^*)]$ is surjective, and if $(x^*(t), y^*(t), z^*(t))$ is normal, then there exist piecewise smooth $\mu^* : I \to R^q$

and $\rho^* : I \to R^n$, *satisfying the following for all* $t \in I$

$$\begin{aligned}
&f_x(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g_x(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \\
&\quad + \rho^*(t)^T h_x(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \\
&\quad + D[f_{\dot{x}}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \\
&\quad + \mu^*(t)^T g_{\dot{x}}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \\
&\quad + \rho^*(t)^T h_{\dot{x}}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))] = 0, \\
&f_y(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g_y(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \\
&\quad + \rho^*(t)^T h_y(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \dot{\rho}^*(t) = 0, \\
&f_z(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g_z(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \\
&\quad + \rho^*(t)^T h_z(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) = 0, \\
&\qquad \mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) = 0, \\
&\qquad\qquad \mu^*(t) \geq 0.
\end{aligned}$$

Proof. It is very similar to that of Theorem 1 of [2]. ∎

Now, the following control dual problems of (MCP) are given

$$\begin{aligned}
\text{(MCD1) maximize } &\bigg(\int_a^b \big[f_1(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \big]\, dt, \ldots, \\
&\int_a^b \big[f_p(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \big]\, dt \bigg)
\end{aligned}$$

$$\begin{aligned}
\text{subject to } \; &u(a) = v(a) = w(a) = 0, \quad u(b) = v(b) = w(b) = 0, \\
&\dot{u}(a) = \dot{u}(b) = 0, \qquad\qquad (4)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i f_{ix}(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\quad + \rho(t)^T h_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\quad + D\bigg[\sum_{i=1}^{p} \lambda_i f_{i\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\quad + \mu(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\quad + \rho(t)^T h_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \bigg] = 0, \quad t \in I \qquad\qquad (5)
\end{aligned}$$

$$\sum_{i=1}^{p} \lambda_i f_{iy}(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g_y(t, u(t), \dot{u}(t), v(t), w(t))$$
$$+ \rho(t)^T h_y(t, u(t), \dot{u}(t), v(t), w(t)) + \dot{\rho}(t) = 0, \quad t \in I \tag{6}$$
$$\sum_{i=1}^{p} \lambda_i f_{iz}(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g_z(t, u(t), \dot{u}(t), v(t), w(t))$$
$$+ \rho(t)^T h_z(t, u(t), \dot{u}(t), v(t), w(t)) = 0, \quad t \in I \tag{7}$$
$$\int_a^b \rho(t)^T \big[h(t, u(t), \dot{u}(t), v(t), w(t)) - \dot{v}(t)\big]\, dt \geq 0, \tag{8}$$
$$\mu(t) \in R_+^q, \quad \rho(t) \in R^n \quad t \in I; \;\; \lambda \in \Lambda^+ \tag{9}$$

and

$$\text{(MCD2) maximize } \left(\int_a^b f_1(t, u(t), \dot{u}(t), v(t), w(t))\, dt, \ldots, \int_a^b f_p(t, u(t), \dot{u}(t), v(t), w(t))\, dt\right)$$

subject to (4), (5), (6), (7), (8), (9),

$$\text{and } \int_a^b \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t))\, dt \geq 0, \tag{10}$$

where (MCD1) is said to be the Wolfe-type vector control dual problem, and (MCD2) be the Mond–Wier-type vector control dual problem.

To deduce our main results, the following definitions are necessary. These definitions generalize those of [3–5, 8].

Definition 3. If there exists a vector function $\eta(t, x(t), \dot{x}(t), y(t), \dot{y}(t), z(t), \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) \in \mathbf{R}_+^n$, with $\eta = 0$ at t if $x(t) = \bar{x}(t)$ or $y(t) = \bar{y}(t)$ or $z(t) = \bar{z}(t)$, and $\xi(t, x(t), \dot{x}(t), y(t), \dot{y}(t), z(t), \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) \in \mathbf{R}^m$ such that for the scalar function $h(t, x(t), \dot{x}(t), y(t), \dot{y}(t), z(t))$ the functional

$$H(x, \dot{x}, y, \dot{y}, z) = \int_a^b h(t, x(t), \dot{x}(t), y(t), \dot{y}(t), z(t))\, dt$$

satisfies

$$H(x, \dot{x}, y, \dot{y}, z) - H(\bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}, \bar{z})$$
$$\geq \int_a^b \{\eta^T[h_x(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))$$
$$+ h_y(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))]$$

$$+ (D\eta)^T \big[h_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) \\ + h_{\dot{y}}(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))\big] \\ + \xi^T h_z(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))\big\}\, dt,$$

then $H(x, \dot{x}, y, \dot{y}, z)$ is said to be invex in $\bar{x}$, $\dot{\bar{x}}$, $\bar{y}$, $\dot{\bar{y}}$, and $\bar{z}$ on I with respect to η and ξ.

DEFINITION 4. If for all $x \in X$, $y \in Y$, and $z \in Z$

$$\int_a^b \{\eta^T [h_x(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) + h_y(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))] \\ + (D\eta)^T [h_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) \\ + h_{\dot{y}}(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))] \\ + \xi^T h_z(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))\}\, dt \geq 0 \\ \Longrightarrow H(x, \dot{x}, y, \dot{y}, z) \geq (>)\, H(\bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}, \bar{z}),$$

then $H(x, \dot{x}, y, \dot{y}, z)$ is said to be (strictly) pseudoinvex in $\bar{x}$, $\dot{\bar{x}}$, $\bar{y}$, $\dot{\bar{y}}$, and $\bar{z}$ on I with respect to η and ξ.

DEFINITION 5. If for all $x \in X$, $y \in Y$, and $z \in Z$

$$H(x, \dot{x}, y, \dot{y}, z) \leq H(\bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}, \bar{z}) \\ \Longrightarrow \int_a^b \{\eta^T [h_x(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) + h_y(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))] \\ + (D\eta)^T [h_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t)) + h_{\dot{y}}(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))] \\ + \xi^T h_z(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t), \bar{z}(t))\}\, dt \leq (<)\, 0,$$

then $H(x, \dot{x}, y, \dot{y}, z)$ is said to be (strictly) quasiinvex in $\bar{x}$, $\dot{\bar{x}}$,$\bar{y}$, $\dot{\bar{y}}$, and $\bar{z}$ on I with respect to η and ξ.

3. DUALITY BETWEEN (MCP) AND (MCD1)

We now give the duality theorems for (MCP) under the above invexity

THEOREM 1 (Weak duality). *Assume that for all feasible solutions* $(x(t), y(t), z(t))$ *of (MCP), and for all feasible solutions* $(u(t), v(t), w(t), \lambda, \mu(t), \rho(t))$ *of (MCD1),* $\int_a^b [f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))]\, dt$ *is invex in u,u̇,v, and w on I with respect to the same* $\eta \in \mathbf{R}^n_+$ *and* $\xi \in R^m$, $i = 1, 2, \ldots, p$; $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ *is also invex in u, u̇, v, and w on I with respect to the above* η *and* ξ. *Then the following inequalities cannot hold simultaneously.*

(1) *For all* $i \in \{1, 2, \dots, p\}$

$$\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt \le \int_a^b \left[f_i(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \right] dt. \tag{11}$$

(2) *For at least one* $j \in \{1, 2, \dots, p\}$

$$\int_a^b f_j(t, x(t), \dot{x}(t), y(t), z(t))\, dt < \int_a^b \left[f_j(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \right] dt. \tag{12}$$

Proof. For each feasible solution $(x(t), y(t), z(t))$ of (MCP) and each feasible solution $(u(t), v(t), w(t), \lambda, \mu(t), \rho(t))$ of (MCD1)

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \int_a^b \left[f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t)) \right] dt \\
&\qquad - \sum_{i=1}^{p} \lambda_i \int_a^b \Big[f_i(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \Big] dt \\
&\ge \sum_{i=1}^{p} \lambda_i \int_a^b \Big\{ \eta^T \big[f_{ix}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + \mu(t)^T g_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + f_{iy}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + \mu(t)^T g_y(t, u(t), \dot{u}(t), v(t), w(t)) \big] \\
&\qquad + (D\eta)^T \big[f_{i\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + \mu(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \big] \\
&\qquad + \xi^T \big[f_{iz}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + \mu(t)^T g_z(t, u(t), \dot{u}(t), v(t), w(t)) \big] \Big\} dt \\
&= \int_a^b \Bigg\{ \eta^T \Bigg[\sum_{i=1}^{p} \lambda_i f_{ix}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + \mu(t)^T g_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad + D\Bigg(\sum_{i=1}^{p} \lambda_i f_{i\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + \mu(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t) \Bigg) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\eta^T\Bigg[\sum_{i=1}^{p}\lambda_i f_{iy}(t,u(t),\dot u(t),v(t),w(t))\\
&\qquad+\mu(t)^T g_y(t,u(t),\dot u(t),v(t),w(t))\Bigg]\\
&\qquad+\xi^T\Bigg[\sum_{i=1}^{p}\lambda_i f_{iz}(t,u(t),\dot u(t),v(t),w(t))\\
&\qquad\qquad+\mu(t)^T g_z(t,u(t),\dot u(t),v(t),w(t))\Bigg]\Bigg\}\,dt\\
&=\int_a^b\big\{\eta^T\big[-\rho(t)^T h_x(t,u(t),\dot u(t),v(t),w(t))\\
&\qquad-D\big(\rho(t)^T h_{\dot x}(t,u(t),\dot u(t),v(t),w(t))\big)\big]\\
&\quad+\eta^T\big[-\rho(t)^T h_y(t,u(t),\dot u(t),v(t),w(t))-\dot\rho(t)\big]\\
&\quad+\xi^T\big[-\rho(t)^T h_z(t,u(t),\dot u(t),v(t),w(t))\big]\big\}\,dt\\
&=-\int_a^b\big\{\eta^T\big[\rho(t)^T h_x(t,u(t),\dot u(t),v(t),w(t))\\
&\qquad+\rho(t)^T h_y(t,u(t),\dot u(t),v(t),w(t))\\
&\qquad+D\big(\rho(t)^T h_{\dot x}(t,u(t),\dot u(t),v(t),w(t))\big)+\dot\rho(t)\big]\\
&\quad+\xi^T\big[\rho(t)^T h_z(t,u(t),\dot u(t),v(t),w(t))\big]\big\}\,dt,
\end{aligned}
\tag{13}
$$

where the first inequality is obtained from the invexity of $\int_a^b[f_i(t,x(t),\dot x(t),y(t),z(t))+\mu(t)^T g(t,x(t),\dot x(t),\ y(t),z(t))]\,dt,\ i=1,2,\ldots,p;$ the second equality is obtained from integrating by parts, when $t=a$ and $t=b$, $x(t)=u(t)$, so it follows that $\eta=0$; the third is obtained from (5)–(7).

By (2) and (8),

$$
\begin{aligned}
&\int_a^b \rho(t)^T\big[h(t,x(t),\dot x(t),y(t),z(t))-\dot y(t)\big]\,dt\\
&\qquad\le\int_a^b \rho(t)^T\big[h(t,u(t),\dot u(t),v(t),w(t))-\dot v(t)\big]\,dt.
\end{aligned}
$$

The invexity of $\int_a^b \rho(t)^T\big[h(t,x(t),\dot x(t),y(t),z(t))-\dot y(t)\big]\,dt$ implies

$$
\begin{aligned}
&\int_a^b\big\{\eta^T\big[\rho(t)^T h_x(t,u(t),\dot u(t),v(t),w(t))\big]\\
&\qquad+(D\eta)^T\big[\rho(t)^T h_{\dot x}(t,u(t),\dot u(t),v(t),w(t))\big]
\end{aligned}
$$

$$+\eta^T\big[\rho(t)^T h_y(t,u(t),\dot{u}(t),v(t),w(t))\big]+(D\eta)^T\big[(-E_{n\times n})\rho(t)\big]$$
$$+\,\xi^T\big[\rho(t)^T h_z(t,u(t),\dot{u}(t),v(t),w(t))\big]\big\}\,dt \le 0, \tag{14}$$

where $E_{n\times n}$ is $n\times n$ unit matrix. Integrating $\int_a^b (D\eta^T)\,[(-E_{n\times n})\rho(t)]\,dt$ by parts

$$\int_a^b (D\eta)^T\big[(-E_{n\times n})\rho(t)\big]\,dt$$
$$= \big[-\eta^T\dot{\rho}(t)\big]\Big|_a^b + \int_a^b \eta^T\dot{\rho}(t)\,dt = \int_a^b \eta^T\dot{\rho}(t)\,dt.$$

From (14), we have

$$\int_a^b \big\{\eta^T\big[\rho(t)^T h_x(t,u(t),\dot{u}(t),v(t),w(t)) + \rho(t)^T h_y(t,u(t),\dot{u}(t),v(t),w(t))$$
$$+\,D\big(\rho(t)^T h_{\dot{x}}(t,u(t),\dot{u}(t),v(t),w(t))\big)+\dot{\rho}(t)\big]$$
$$+\,\xi^T\big[\rho(t)^T h_z(t,u(t),\dot{u}(t),v(t),w(t))\big]\big\}\,dt \le 0. \tag{15}$$

So, from (13) we get

$$\sum_{i=1}^p \lambda_i \int_a^b \big[f_i(t,x(t),\dot{x}(t),y(t),z(t))+\mu(t)^T g(t,x(t),\dot{x}(t),y(t),z(t))\big]dt$$
$$\ge \sum_{i=1}^p \lambda_i \int_a^b \big[f_i(t,u(t),\dot{u}(t),v(t),w(t))+\mu(t)^T g(t,u(t),\dot{u}(t),v(t),w(t))\big]dt.$$

By (3) and (9), we obtain

$$\sum_{i=1}^p \lambda_i \int_a^b f_i(t,x(t),\dot{x}(t),y(t),z(t))\,dt$$
$$\ge \sum_{i=1}^p \lambda_i \int_a^b \big[f_i(t,u(t),\dot{u}(t),v(t),w(t))+\mu(t)^T g(t,u(t),\dot{u}(t),v(t),w(t))\big]dt.$$

The proof is complete. ∎

Remark 1. From the proof of Theorem 1, we can also obtain that (11) and (12) cannot hold simultaneously if one of the following conditions holds:

(1) $\int_a^b [f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))]dt$ is invex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ, $i = 1, 2, \ldots, p$; $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ is quasiinvex in $x, \dot{x}, y$, and z with respect to η and ξ.

(2) $\lambda \in \Lambda^+$, $\int_a^b [f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))]\, dt$ is pseudoinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ, $i = 1, 2, \ldots, p$; $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ is invex or quasiinvex $x, \dot{x}, y$, and z with respect to η and ξ.

(3) $\lambda \in \Lambda^+$, $\int_a^b [f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))]\, dt$ is strictly quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ, $i = 1, 2, \ldots, p$; $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ is quasiinvex $x, \dot{x}, y$, and z with respect to η and ξ.

(4) $\int_a^b \left[\sum_{i=1}^p f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\right] dt$ and $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ is strictly quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ.

(5) $\int_a^b \left[\sum_{i=1}^p f_i(t, x(t), \dot{x}(t), y(t), z(t)) + \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\right] dt$ is quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ; $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ is strictly pseudoinvex in $x, \dot{x}, y$, and z on I with respect to η and ξ. ∎

THEOREM 2 (Strong duality). *Assume that the invexities of Theorem* 1 *are satisfied. Let* $(x^*(t), y^*(t), x^*(t))$ *be a properly efficient solution of* (*MCP*). *If* $(x^*(t), y^*(t), z^*(t))$ *satisfies the constraint qualification of Lemma* 2 *for the following problem for some* $\lambda^* \in \Lambda^+$

$$(P_{\lambda^*}) \quad \textit{minimize} \ \sum_{i=1}^p \lambda_i^* \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt$$

$$\textit{subject to } (1), (2), \textit{ and } (3),$$

then there exist piecewise smooth $\mu^* : I \to R^q$, $\rho^* : I \to R^n$ *such that* $(x^*(t), y^*(t), z^*(t), \lambda^*, \mu^*(t), \rho^*(t))$ *is a properly efficient solution of* (*MCD*1).

Proof. Now that $(x^*(t), y^*(t), z^*(t))$ is a properly efficient solution of (MCP), from Lemma 1, $(x^*(t), y^*(t), z^*(t))$ solves (P_{λ^*}) for some $\lambda^* \in \Lambda^+$. As $(x^*(t), y^*(t), z^*(t))$ satisfies the constraint qualification for (P_{λ^*}) for some $\lambda^* \in \Lambda^+$, it follows from Lemma 2 that there exist piecewise smooth $\mu^* : I \to R^q$ and $\rho^* : I \to R^n$, satisfying the following for all $t \in I$

$$\sum_{i=1}^p \lambda_i^* f_{ix}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g_x(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$
$$+ \rho^*(t)^T h_x(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$

$$+D\left[\sum_{i=1}^{p}\lambda_i^* f_{i\dot{x}}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\right.$$
$$+\mu^*(t)^T g_{\dot{x}}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$
$$\left.+\rho^*(t)^T h_{\dot{x}}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\right] = 0,$$

$$\sum_{i=1}^{p}\lambda_i^* f_{iy}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g_y(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$
$$+\rho^*(t)^T h_y(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \dot{\rho}^*(t) = 0,$$

$$\sum_{i=1}^{p}\lambda_i^* f_{iz}(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g_z(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$
$$+\rho^*(t)^T h_z(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) = 0,$$
$$\mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) = 0,$$
$$\mu^*(t) \geq 0.$$

From (2), $\int_a^b \rho^*(t)^T[h(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) - \dot{y}^*(t)]\,dt = 0$. So, $(x^*(t), y^*(t), z^*(t), \lambda^*, \mu^*(t), \rho^*(t))$ is a feasible solution of (MCD1).

If $(x^*(t), y^*(t), z^*(t), \lambda^*, \mu^*(t), \rho^*(t))$ is not an efficient feasible solution of (MCD1), then there exists a feasible solution $(u(t), v(t), w(t), \lambda, \mu(t), \rho(t))$ of (MCD1) such that for all $i \in \{1, 2, \ldots, p\}$

$$\int_a^b \Big[f_i(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t))\Big]\,dt$$
$$\geq \int_a^b \Big[f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$
$$+\mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\Big]\,dt\ ,$$

and for at least one $j \in \{1, 2, \ldots, p\}$

$$\int_a^b [f_j(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t))]\,dt$$
$$> \int_a^b [f_j(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))$$
$$+\mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))]\,dt.$$

Since $\mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) = 0$, we have for all $i \in \{1, 2, \ldots, p\}$

$$\int_a^b \Big[f_i(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \Big] dt \geq \int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\, dt,$$

and for at least one $j \in \{1, 2, \ldots, p\}$

$$\int_a^b \Big[f_j(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \Big] dt > \int_a^b f_j(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\, dt,$$

which contradicts the conclusion of Theorem 1.

Now, we assume that $(x^*(t), y^*(t), z^*(t), \lambda^*, \mu^*(t), \rho^*(t))$ is not a properly efficient feasible solution of (MCD1), i.e., there exists a feasible solution $(u(t), v(t), w(t), \lambda, \mu(t), \rho(t))$ of (MCD1) such that for some $i \in \{1, 2, \ldots, p\}$ and any real $M > 0$

$$\int_a^b \Big[f_i(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \Big] dt - \int_a^b \big[f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) + \mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) \big] dt > M.$$

Again from $\mu^*(t)^T g(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t)) = 0$, we have

$$\sum_{i=1}^{p} \lambda_i^* \int_a^b \big[f_i(t, u(t), \dot{u}(t), v(t), w(t)) + \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t)) \big] dt - \sum_{i=1}^{p} \lambda_i^* \int_a^b f_i(t, x^*(t), \dot{x}^*(t), y^*(t), z^*(t))\, dt > M,$$

which contradicts Theorem 1 again and the proof is complete. ∎

THEOREM 3 (Converse duality). *Assume that weak duality (Theorem 1) holds between (MCP) and (MCD1). If* $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ *is feasible for (MCP) and* $(\bar{u}(t), \bar{v}(t), \bar{w}(t), \bar{\lambda}, \bar{\mu}(t), \bar{\rho}(t))$ *is feasible for (MCD1) with* $\bar{\mu}(t)^T g(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t)) = 0$, *then* $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ *is properly efficient for (MCP) and* $(\bar{u}(t), \bar{v}(t), \bar{w}(t), \bar{\lambda}, \bar{\mu}(t), \bar{\rho}(t))$ *is properly efficient for (MCD1).*

Proof. Suppose that $(\bar{u}(t), \bar{v}(t), \bar{w}(t)$ is not efficient for (MCP), then there exists some feasible solution $(x(t), y(t), z(t))$ of (MCP) such that for all $i \in \{1, 2, \dots, p\}$

$$\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt \le \int_a^b f_i(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t))\, dt,$$

and for some $j \in \{1, 2, \dots, p\}$

$$\int_a^b f_j(t, x(t), \dot{x}(t), y(t), z(t))\, dt < \int_a^b f_j(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t))\, dt.$$

Since $\bar{\mu}(t)^T g(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t)) = 0$, we have for all $i \in \{1, 2, \dots, p\}$

$$\begin{aligned}
&\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt \\
&\quad \le \int_a^b \left[f_i(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t)) + \bar{\mu}(t)^T g(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t))\right] dt,
\end{aligned}$$

and for some $j \in \{1, 2, \dots, p\}$

$$\begin{aligned}
&\int_a^b f_j(t, x(t), \dot{x}(t), y(t), z(t))\, dt \\
&\quad \le \int_a^b \left[f_j(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t)) + \bar{\mu}(t)^T g(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t))\right] dt,
\end{aligned}$$

which contradicts the weak duality. Hence $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ is efficient for (MCP). It is similar to the proof of Theorem 2 that $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ is also a properly efficient for (MCP).

Similarly, we deduce that $(\bar{u}(t), \bar{v}(t), \bar{w}(t), \bar{\lambda}, \bar{\mu}(t), \bar{\rho}(t))$ is also properly efficient for (MCD1). ∎

4. DUALITY BETWEEN (MCP) AND (MCD2)

In this section, we discuss the duality between (MCP) and (MCD2). The proof is similar to that of Theorems 1–3.

THEOREM 4 (Weak duality). *Assume that for all feasible solutions $(x(t), y(t), z(t))$ of (MCP), and for all feasible solutions $(u(t), v(t), w(t), \lambda, \mu(t), \rho(t))$ of (MCD2), $\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))dt$, $i = 1, 2, \dots, p$; $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\, dt$ and $\int_a^b \rho(t)^T [h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$ are invex in u,$\dot{u}$,v, and w on I with respect to the same $\eta \in \mathbf{R}^n_+$ and $\xi \in R^m$. Then the following inequalities cannot hold simultaneously.*

(1) *For all $i \in \{1, 2, \dots, p\}$*

$$\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt \le \int_a^b f_i(t, u(t), \dot{u}(t), v(t), w(t))\, dt, \quad (16)$$

(2) *For at least one* $j \in \{1, 2, \ldots, p\}$

$$\int_a^b f_j(t, x(t), \dot{x}(t), y(t), z(t))\,dt < \int_a^b f_j(t, u(t), \dot{u}(t), v(t), w(t))\,dt. \tag{17}$$

Proof. For each feasible solution $(x(t), y(t), z(t))$ of (MCP) and each feasible solution $(u(t), v(t), w(t), \lambda, \mu(t), \rho(t))$ of (MCD2), we have

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt \\
&\qquad - \sum_{i=1}^{p} \lambda_i \int_a^b f_i(t, u(t), \dot{u}(t), v(t), w(t))\,dt \\
&\qquad \geq \sum_{i=1}^{p} \lambda_i \int_a^b \big\{\eta^T \big[f_{ix}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + f_{iy}(t, u(t), \dot{u}(t), v(t), w(t))\big] \\
&\qquad\qquad + (D\,\eta)^T f_{i\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + \xi^T f_{iz}(t, u(t), \dot{u}(t), v(t), w(t)) \big\}\,dt \\
&\qquad = \int_a^b \Bigg\{ \eta^T \Bigg[\sum_{i=1}^{p} \lambda_i f_{ix}(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + D\Bigg(\sum_{i=1}^{p} \lambda_i f_{i\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t))\Bigg)\Bigg] \\
&\qquad\quad + \eta^T \Bigg[\sum_{i=1}^{p} \lambda_i f_{iy}(t, u(t), \dot{u}(t), v(t), w(t))\Bigg] \\
&\qquad\quad + \xi^T \Bigg[\sum_{i=1}^{p} \lambda_i f_{iz}(t, u(t), \dot{u}(t), v(t), w(t))\Bigg]\Bigg\}\,dt \\
&\qquad = \int_a^b \big\{\eta^T \big[-\mu(t)^T g_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad - D\big(\mu(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t))\big) \\
&\qquad\qquad - \rho(t)^T h_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad - D\big(\rho(t)^T h_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t))\big)\big] \\
&\qquad\quad + \eta^T \big[-\mu(t)^T g_y(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad - \rho(t)^T h_y(t, u(t), \dot{u}(t), v(t), w(t)) - \dot{\rho}(t)\big]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \xi^T \big[-\mu(t)^T g_z(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad - \rho(t)^T h_z(t, u(t), \dot{u}(t), v(t), w(t)) \big] \big\} \, dt \\
&= - \int_a^b \big\{ \eta^T \big[\mu(t)^T g_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + D\big(\mu(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t))\big) \\
&\qquad\qquad + \mu(t)^T g_y(t, u(t), \dot{u}(t), v(t), w(t)) \big] \\
&\qquad\qquad + \xi^T \big[\mu(t)^T g_z(t, u(t), \dot{u}(t), v(t), w(t)) \big] \big\} \, dt, \\
&\quad - \int_a^b \big\{ \eta^T \big[\rho(t)^T h_x(t, u(t), \dot{u}(t), v(t), w(t)) \\
&\qquad\qquad + D\big(\rho(t)^T h_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t))\big) \\
&\qquad\qquad + \rho(t)^T h_y(t, u(t), \dot{u}(t), v(t), w(t)) + \dot{\rho}(t) \big] \\
&\qquad\qquad + \xi^T \big[\rho(t)^T h_z(t, u(t), \dot{u}(t), v(t), w(t)) \big] \big\} \, dt, \qquad (18)
\end{aligned}
$$

where the first inequality is obtained from the invexity of $\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt \;\; i = 1, 2, \ldots, p$; the second equality is obtained from integrating by parts, when $t = a$ and $t = b$, $x(t) = u(t)$, so it follows that $\eta = 0$; the third is obtained from (5)–(7).

By (3), (9), and (10),

$$
\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\, dt \le \int_a^b \mu(t)^T g(t, u(t), \dot{u}(t), v(t), w(t))\, dt.
$$

The invexity of $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\, dt$ implies

$$
\begin{aligned}
&\int_a^b \big\{ \eta^T \big[\mu(t)^T g_x(t, u(t), \dot{u}(t), v(t), w(t)) \big] \\
&\qquad + (D\eta)^T \big[\mu(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t), v(t), w(t)) \big] \\
&\qquad + \eta^T \big[\mu(t)^T g_y(t, u(t), \dot{u}(t), v(t), w(t)) \big] \\
&\qquad + \xi^T \big[\mu(t)^T g_z(t, u(t), \dot{u}(t), v(t), w(t)) \big] \big\} \, dt \le 0. \qquad (19)
\end{aligned}
$$

By the invexity of $\int_a^b \rho(t)^T [\, h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\, dt$, we have (15). So, from (15), (18), and (19)

$$
\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\, dt \\
&\qquad \ge \sum_{i=1}^{p} \lambda_i \int_a^b f_i(t, u(t), \dot{u}(t), v(t), w(t)) \,]\, dt.
\end{aligned}
$$

The proof is finished. ∎

Remark 2. From the proof of Theorem 4, (16) and (17) cannot hold simultaneously if one of the following conditions holds:

(1) $\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ is invex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ, $i = 1, 2, \ldots, p$; $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ and $\int_a^b \rho(t)^T[h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\,dt$ are quasiinvex in $x, \dot{x}, y$, and z with respect to η and ξ.

(2) $\lambda \in \Lambda^+$, $\int_a^b \sum_{i=1}^p f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ is pseudoinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ, $i = 1, 2, \ldots, p$; $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ and $\int_a^b \rho(t)^T[h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\,dt$ are invex or quasiinvex $x, \dot{x}, y$, and z with respect to η and ξ.

(3) $\lambda \in \Lambda^+$, $\int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ is strictly quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ, $i = 1, 2, \ldots, p$; $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ and $\int_a^b \rho(t)^T[h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\,dt$ are quasiinvex $x, \dot{x}, y$, and z with respect to η and ξ.

(4) $\lambda \in \Lambda^+$, $\int_a^b \sum_{i=1}^p f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ is quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ; $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ and $\int_a^b \rho(t)^T[h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\,dt$ are strictly quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ.

(5) $\lambda \in \Lambda^+$, $\int_a^b \sum_{i=1}^p f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ is quasiinvex in $x, \dot{x}, y$, and z on I with respect to the same η and ξ; $\int_a^b \mu(t)^T g(t, x(t), \dot{x}(t), y(t), z(t))\,dt$ and $\int_a^b \rho(t)^T[h(t, x(t), \dot{x}(t), y(t), z(t)) - \dot{y}(t)]\,dt$ are strictly pseudoinvex in $x, \dot{x}, y$, and z on I with respect to η and ξ. ∎

The proof of the next two theorems is similar to that of Theorems 2 and 3.

THEOREM 5 (Strong duality). *Assume that the invexities of Theorem* 4 *are satisfied. Let* $(x^*(t), y^*(t), x^*(t))$ *be a properly efficient solution of* (*MCP*). *If* $(x^*(t), y^*(t), z^*(t))$ *satisfies the constraint qualification of Lemma* 2 *for the following problem for some* $\lambda^* \in \Lambda^+$

$$(P_{\lambda^*})\ \textit{minimize}\ \sum_{i=1}^p \lambda_i^* \int_a^b f_i(t, x(t), \dot{x}(t), y(t), z(t))\,dt$$

$$\textit{subject to}\ (1), (2),\ \textit{and}\ (3),$$

then there exist piecewise smooth $\mu^* : I \to R^q, \rho^* : I \to R^n$ *such that* $(x^*(t), y^*(t), z^*(t), \lambda^*, \mu^*(t), \rho^*(t))$ *is a properly efficient solution of* (*MCD*2). ∎

THEOREM 6 (Converse duality). *Assume that weak duality* (*Theorem* 4) *holds between* (*MCP*) *and* (*MCD*2). *If* $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ *is feasible for* (*MCP*) *and* $(\bar{u}(t), \bar{v}(t), \bar{w}(t), \bar{\lambda}, \bar{\mu}(t), \bar{\rho}(t))$ *is feasible for* (*MCD*2) *with* $\bar{\mu}(t)^T g(t, \bar{u}(t), \dot{\bar{u}}(t), \bar{v}(t), \bar{w}(t)) = 0$, *then* $(\bar{u}(t), \bar{v}(t), \bar{w}(t))$ *is properly efficient for* (*MCP*) *and* $(\bar{u}(t), \bar{v}(t), \bar{w}(t), \bar{\lambda}, \bar{\mu}(t), \bar{\rho}(t))$ *is properly efficient for* (*MCD*2). ∎

In the above, we have discussed the duality between (MCP) and (MCD1) (MCD2). Here our conclusions include some results as found in [2, 3, 7, 8].

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